The depth-averaged Mixing Length turbulence model for Telemac-2D

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***Abstract*—In this paper the depth-averaged Mixing Length turbulence model and its implementation in Telemac-2D is presented. The new turbulence model is verified and validated by means of a laboratory experiment concerning the flow around a spur-dyke. The experiment is well suitable for testing the Mixing Length model since in the region of a groyne-like structure significant horizontal flow velocity gradients with possible recirculation flows prevail, which in turn have an influence on the turbulence production, the computed turbulent eddy viscosity and the resulting velocity distribution. Additionally to these laboratory measurements, the implemented Mixing Length model is compared to the depth-averaged k-ε turbulence model. The validation reveals the correct implementation of the turbulence model and its applicability for open channel flow computations.**

I. MOTIVATION

In flows with high transverse velocity gradients, e.g. flows around structures, strong recirculation flows or flows in reservoirs, the influence of the horizontal velocity gradients on the turbulence production can be significant. In such cases the transverse shear may be the dominant turbulence generation mechanism in contrast to e.g. straight river applications where usually most of the river turbulence is generated by bed friction. Hence the main idea is to combine the depth-averaged parabolic eddy viscosity model with the Prandtl's mixing length theory for the horizontal in order to account for both the vertical and horizontal turbulence production. The resulting depth-averaged Mixing Length turbulence model forms a zero-equation turbulence model, which, as per this definition, doesn't account for any transport processes.

The main characteristic of the Mixing Length model implemented in Telemac-2D is that it accounts for the physical influence of the local horizontal velocity gradients on the turbulent eddy viscosity to be computed. The model yields or tends to the parabolic eddy viscosity model if the horizontal depth-averaged velocity gradients vanish or if the turbulence is mainly produced by bed friction, respectively.

II. EDDY VISCOSITY CONCEPT IN TELEMAC-2D

Telemac-2D solves the depth-averaged Saint-Venant equations in two dimensions (www.opentelemac.org). The turbulent diffusion by means of the depth-averaged Reynolds stresses appearing in the depth-averaged momentum equations are determined by the Boussinesq's eddy viscosity assumption [1]. This hypothesis assumes that, in analogy to the viscous stresses in laminar flows, the turbulent stresses are proportional to the mean velocity gradients or in other words, the momentum transfer caused by turbulent eddies can be modelled with an eddy viscosity. The turbulent eddy viscosity ν_t is not a fluid property but strongly depends on the local state of turbulence and may vary largely in time and space. The role of the turbulence model is to determine the turbulent viscosity ν_t and its spatial and time dependent distribution in a model domain.

III. THE DEPTH-AVERAGED MIXING LENGTH MODEL

In the depth-averaged Mixing Length model the total turbulent viscosity ν_t is split in a vertical component ν_t^V and a horizontal component ν_t^H [2]:

$$\nu_t = \sqrt{(\nu_t^V)^2 + (\nu_t^H)^2} \tag{1}$$

A. *Calculation of the vertical eddy viscosity*

The vertical eddy viscosity ν_t^V is computed by means of the depth-averaged parabolic eddy viscosity model in which the vertical viscosity is generated by bed friction. This model implies a perfect balance between hydrostatic pressure gradient and vertical shear stress. With the assumption of two-dimensional flow and a logarithmic velocity profile along the water depth the vertical eddy viscosity follows a parabolic profile along the depth. Starting with the Prandtl's Mixing Length hypothesis the eddy viscosity along the water depth $\nu_{t,z}$ is related to the mean velocity gradient and the mixing length:

$$\nu_{t,z} = l_m^2 \left|\frac{\partial u}{\partial z}\right| \tag{2}$$

where u is the mean flow velocity, z is the vertical coordinate and l_m is the mixing length. Assuming a logarithmic velocity profile the vertical velocity gradient is:

$$\frac{\partial u}{\partial z} = \frac{U^*}{\kappa z} \tag{3}$$

where U^* is the shear velocity and κ is the von Kármán constant equal to 0.4.

The mixing length distribution $l_m(z)$ along the water depth is given by [3]:

$$l_m(z) = \kappa z \sqrt{1 - \frac{z}{h}} \tag{4}$$

where h is the water depth.

Substituting (3) and (4) into (2) and integrating (2) over the water depth the depth-averaged vertical eddy viscosity ν_t^V is obtained as:

$$\nu_t^V = \frac{1}{h}\kappa U^* \int_0^h z\left(1 - \frac{z}{h}\right) dz = \frac{1}{6}\kappa U^* h = \alpha_t\, U^* h \tag{5}$$

The basic assumption of the depth-averaged parabolic eddy viscosity model is that in open channel flow the turbulence is mainly generated by bed friction in that the depth mean turbulent viscosity is correlated with the shear velocity U^* and the water depth h. The theoretical proportionality constant $\kappa/6 = \alpha_t$ in (5) is valid only for infinitely wide channels and doesn't account for anisotropic structures of turbulence in horizontal and vertical directions as well as for the transversal or longitudinal dispersion. So for most of the 2D depth-averaged applications this constant can be considered as too low. Elder [4] and later Fischer et al. [5] developed, based on the equation and experiments in laboratory channels and natural streams, dispersion equations for the transport of substances in natural streams and determined higher values for the proportionality constant α_t. Fischer et al. [5] propose that for transverse turbulent dispersion α_t is about 0.15 in laboratory channels and 0.6 in irregular natural streams with weak meanders. Wu et al. [6] compare five depth-averaged turbulence models in the simulation of flows around a spur-dyke, in a sudden-expanded flume and in two natural rivers. They apply values for α_t in from 0.6 to 1.0. Vionnet et al. [7] in turn, use in their numerical models values in the range of $\kappa/6$ to 0.3. Jia and Wang [8] employ in their 2D depth-averaged numerical model the coefficient $\alpha_t = A \cdot \kappa/6$ with A as calibration parameter for which they recommend values in the range of 1 to 10. Steffler and Blackburn [9] in the River2D model for α_t use a default value of 0.5 and indicate values from 0.2 to 1.0 as a reasonable range. As it can be seen from these elaborations the proportionality coefficient α_t has to be considered as a calibration coefficient. For the implementation in Telemac-2D the theoretical constant in (5) has been replaced by a selectable empirical calibration coefficient α_t (with $\alpha_t = 1/6\,\kappa \approx 0.067$ as default value).

B. *Calculation of the horizontal eddy viscosity*

The horizontal eddy viscosity ν_t^H is computed according to the Prandtl's mixing length theory by means of the depth-averaged horizontal mixing length l_m and the horizontal mean strain-rate tensor S_{ij}:

$$\nu_t^H = l_m^2 \sqrt{2 S_{ij} S_{ij}} \tag{6}$$

The horizontal mean strain-rate tensor S_{ij}

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{7}$$

is computed by means of the depth-averaged velocity derivatives, written in Cartesian coordinates:

$$2 S_{ij} S_{ij} = 2\left(\frac{\partial U}{\partial x}\right)^2 + 2\left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)^2 \tag{8}$$

The depth-averaged mixing length l_m is calculated by integrating equation (4) over the water depth:

$$l_m = \frac{1}{h}\kappa \int_0^h z\sqrt{1 - \frac{z}{h}}\, dz = \frac{4}{15}\kappa h \tag{9}$$

Inserting (8) and (9) into (6) yields the horizontal turbulent viscosity ν_t^H due to horizontal shear:

$$\nu_t^H = \left(\frac{4}{15}\kappa h\right)^2 \sqrt{2\left(\frac{\partial U}{\partial x}\right)^2 + 2\left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)^2} \tag{10}$$

In (9) it is assumed that the mixing length l_m depends on the water depth h which restricts the size of the turbulent eddies. However the assumption l_m as a function of the water depth can lead to an underestimation of ν_t^H since l_m may be larger than the water depth [2]. The dependence of the mixing length l_m on the water depth can be seen as a weakness of the Mixing Length model. Hence for the implementation in Telemac-2D the theoretical constant $4/15\,\kappa$ in (10) has been replaced by a selectable empirical calibration coefficient C_l (with $C_l = 4/15\,\kappa \approx 0.107$ as default value).

The literature research regarding the use of the horizontal Mixing Length model and the related choice of the C_l coefficient in typical open channel flow simulations has not given that many results. Wu et al. [6] in the before mentioned four case studies for C_l use values from 0.16 to 0.48. Steffler and Blackburn [9] recommend a C_l coefficient of 0.1 as a typical value which corresponds to the theoretical coefficient. However they point out that depending on the type of flow the factor C_l may be adjusted. Stansby [10] validated a three-dimensional numerical model against the experimental data for shallow wakes of a conical island. He proposed a two-mixing-length, eddy-viscosity turbulence model with a vertical mixing length of classical Prandtl form and a horizontal mixing length. Stansby estimated the vertical mixing length l_v to be equal to $0.09h$ assuming a boundary layer thickness of $\delta = 0.2h$. He assumed the horizontal mixing length l_h to be a multiple β of the vertical mixing length l_v. Stansby tested this formulation for the replication of either eddy formation or stable wake. He reports good predictions when using a β value equal to 6 which yields a horizontal mixing length l_m of about half the water depth with $C_l \approx 0.5$. Stansby in [11] reduced the 3D approach to the depth-averaged form and investigated the same case by means of a 2D depth-averaged numerical model. By using β equal to 6 the prediction of stable wakes was poor. However, when vortex shedding was prominent, the 2D and 3D model wake structures were similar. Chini and Stansby in [12] implemented the two-mixing-length eddy-viscosity turbulence model into the 3D numerical model Telemac-3D. They tested the model against two datasets. The first case was the flow around a conical island with associated wake patterns. The second case was the tidal flow around a headland. Based on Stansby's findings [10] a ratio of $\beta = 6$ between the horizontal mixing length and the vertical mixing length was applied. In both the case studies Telemac-3D combined with the two mixing length eddy viscosity model could replicate the experimental results. Apparently the approach proposed by Chini and Stansby [12] didn't find the way into an official release of Telemac-3D.

IV. THE DEPTH-AVERAGED MIXING LENGTH MODEL IN TELEMAC-2D

The combination of the parabolic eddy viscosity model (5) and the horizontal Mixing Length model (10) yields finally the depth-averaged Mixing Length model implemented in Telemac-2D in which the eddy viscosity coefficient is composed of three components: a constant, a bed shear generated term and a transverse shear generated term:

$$\nu_t = \nu_{t,c} + \sqrt{(\alpha_t U^* h\,)^2 + \left[(C_l h)^2 \sqrt{2 S_{ij} S_{ij}}\right]^2} \qquad (11)$$

where $\nu_{t,c}$ is the constant eddy viscosity coefficient (keyword: velocity diffusivity) with the default value in Telemac-2D of 1.E-6 m²/s.

Near the wall the damping effect of the wall on the turbulence may be important and thus the relation for the mixing length in (9) could produce too high turbulent viscosities in the wall region. For the mesh nodes near the wall instead of using the water depth as the length scale the distance from the nodes to the wall $dist_{wall}$ should be used. Two different approaches have been tested for the near-wall treatment, namely the method by Jia and Wang [8] and the method by Cea et al. [2]. Both the approaches compute almost identical turbulent viscosity values at the wall boundary nodes. Therefore as limiter for the mixing length $l_{m,w}$ at the wall boundary nodes the simpler method by Cea et al. has been kept:

$$l_{m,w} = min\ C_l\big(h,\ dist_{wall}\big) \qquad (12)$$

The Mixing Length model is activated in the steering file by the keyword TURBULENCE MODEL = 5. The calibration coefficients α_t and C_l may be changed in the subroutine mixlength.f.

V. NUMERICAL VALIDATION: FLOW AROUND A SPUR-DYKE

The turbulence model implemented in Telemac-2D is verified and validated by means of a laboratory experiment concerning the simulation of the flow around a spur-dyke [13]. Rajaratnam and Nwachukwu [13] measured the flow velocities around a spur-dyke in a laboratory flume. The experiment is well suitable for testing the Mixing Length model since in the region of a groyne-like structure significant horizontal flow velocity gradients with possible recirculation flows prevail, which in turn have an influence on the turbulence production, the computed turbulent eddy viscosity and the resulting velocity distribution.

In the validation process it is not intended to perform a sensitivity analysis by varying some physical and numerical parameters with the objective of matching the experimental results. Additionally to the measurements, also the numerical results of the simulation with the depth-averaged k-ε turbulence model are compared. This laboratory experiment has been used as comparative test also by other depth-averaged numerical models like the CCHED2D model [8] and the Coastal Modeling System CMS [14].

A. *Experimental setup*

The experiments were conducted in a straight tilting rectangular flume with the dimensions: 37 m long, 0.91 m wide and 0.76 m deep. The test reach was located in the downstream half of the flume. Rajaratnam and Nwachukwu carried out 13 different experiments by varying the length or the shape of the spur-dyke, the water depth and the bed roughness. For the validation conducted here the experimental run A1 is used.

In experimental run A1 the spur-dyke was made by a 3 mm thin and 0.152 m long aluminium plate projecting perpendicular to the vertical side wall. The flow discharge was 0.0453 m³/s and the approach flow depth was 0.189 m. The flume bed and sides were hydraulically smooth. The flume was inclined to establish uniform flow conditions.

Rajaratnam and Nwachukwu measured the velocity profiles along four cross sections in the locations x/b = 2, 4, 6 and 8, with x starting at the spur-dyke station and b the spur-dyke length (0.152 m). The flow velocities were measured at two vertical levels z/h=0.03 and z/h=0.85. In the experiment the reattachment length of the eddy zone downstream of the spur-dyke was found to be approximately 12b.

B. *Numerical setup*

The computational domain covers 10 m of the flume length. A horizontal flume bed is assumed. The mesh consists of 8780 nodes and 17020 triangular elements with maximal edge lengths of 0.08 m. In the region of the spur-dyke and in the recirculation zone behind the structure a higher mesh resolution with minimal edge lengths of about 0.015 m is used. The spur-dyke is placed 4 m downstream of the inlet and perpendicular to the right wall.

Accordingly to the experiment at the upstream boundary an inflow discharge of 0.0453 m/³s and at the outflow boundary a constant flow depth of 0.189 m are specified. The

Strickler roughness coefficient is set to be equal to 90 m$^{1/3}$/s for the whole domain. For the side walls a fully slip condition is applied. As advection scheme for the flow velocity the explicit MURD scheme is used. In the simulation with the k-ε turbulence model the method of characteristics is used for the advective transport of the turbulent kinetic energy k and the turbulent dissipation ε. In the case of the Mixing Length model the default values for α_t and C_l equal to 0.067 and 0.107, respectively, are applied. A simulation time step of 0.02 seconds is used and the simulation is run until a steady state flow field is reached.

C. Numerical results

The evaluation of the depth-averaged Mixing Length model is shown in Fig. 1 by means of the computed turbulent eddy viscosity ν_t. From a verification point of view the model doesn't produce any unphysical low or high spikes and the spatial distribution is reasonable. Near the spur-dyke where higher velocity gradients prevail the eddy viscosity is accordingly higher than in the surrounding area. Clearly visible is the operation of the limiter for the mixing length l_m in that the eddy viscosity is reduced near the side walls of the flume and the spur-dyke. The comparison with the k-ε turbulence model (Fig. 1) shows that both turbulence models in terms of the computed eddy viscosity behave quite differently. The Mixing Length model gives higher turbulent diffusion near the head of the spur dyke whereas the k-ε model gives much higher eddy viscosity values downstream of the spur dyke.

The numerical results in terms of velocity distribution and the location of the four cross sections are shown in Fig. 2. Both the turbulence models are able to produce the backward-flow region behind the groyne. The Mixing Length model computes a larger recirculation zone downstream of the spur-dyke compared to the k-ε model. In comparison to the measured reattachment length the Mixing Length model (deployed with the standard parameters) slightly overpredicts while the k-ε model underpredicts the recirculation length. In the other regions upstream and downstream of the spur-dyke both the turbulence models produce very similar velocity distributions, which confirms also the correct implementation of the depth-averaged Mixing Length model in Telemac-2D.

For the comparison of the measured velocity profiles with the Telemac-2D results the data measured at level z/h=0.85 are used. Fig. 3 shows the measured and the simulated velocities in x-direction in the four cross sections. The measured data reveal significant negative velocities near the wall and the maximum positive velocities arising just outside the shear layer in all the cross sections. In the main flow region the velocity distribution is almost uniform. Compared to the measurements both the turbulence models provide good predictions of the velocity distributions in the four cross sections. However the Mixing Length model performs better, especially in the cross sections x=6b and x=8b, where the k-ε model largely underpredicts the magnitude of the negative velocities near the wall. It also computes too low flow velocities in the main flow region where the Mixing Length model gives good results. For the quantitative assessment Table 1 lists the root-mean-square error (RMSE) between the measured and the simulated velocities in the four cross sections. The RMSE values show the almost similar performance of both the turbulence models in cross section x=2b. With increasing distance from the spur-dyke x=2b, 4b and 8b the RMSE indicates a considerably higher agreement between the Mixing Length model and the measurements.

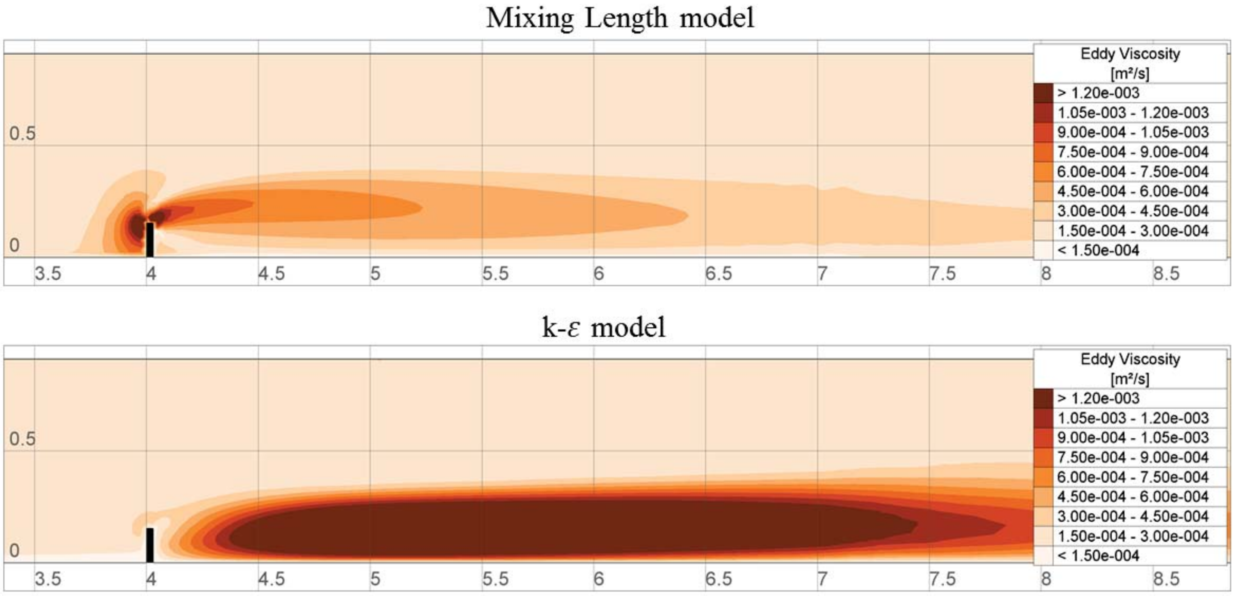

Figure 1. Plan view, computed turbulent eddy viscosities by the Mixing Length model and the k- ε model

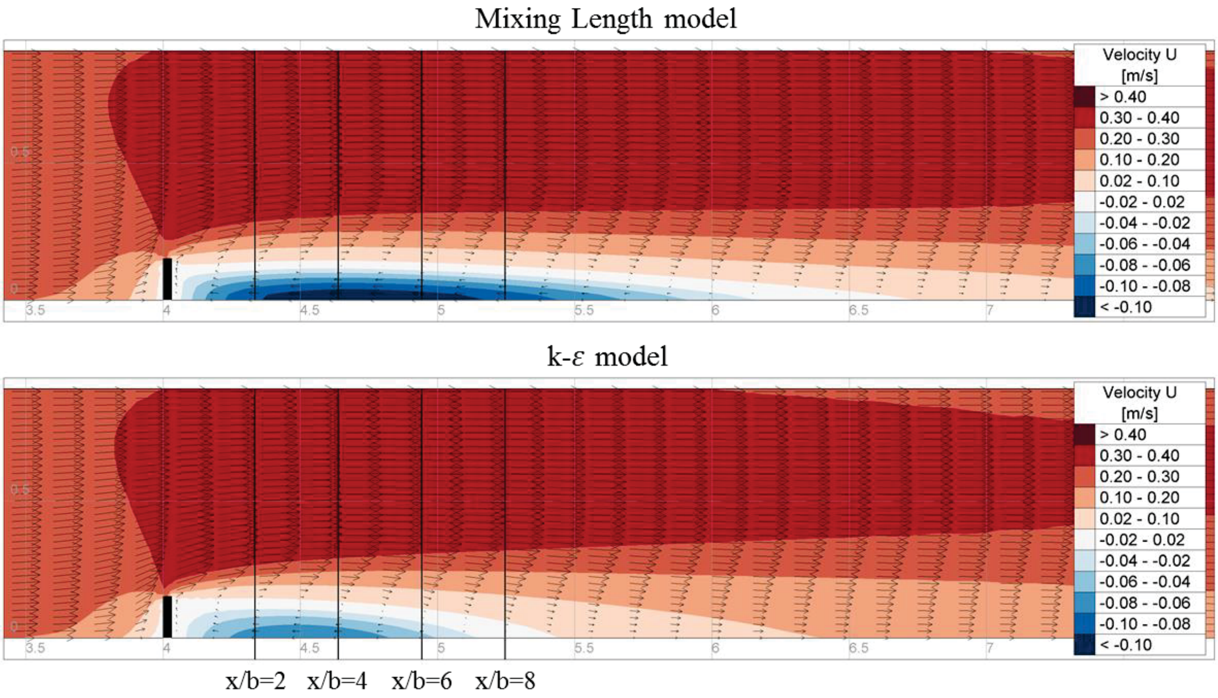

Figure 2. Plan view, computed flow velocities by the Mixing Length model and the k- ε model

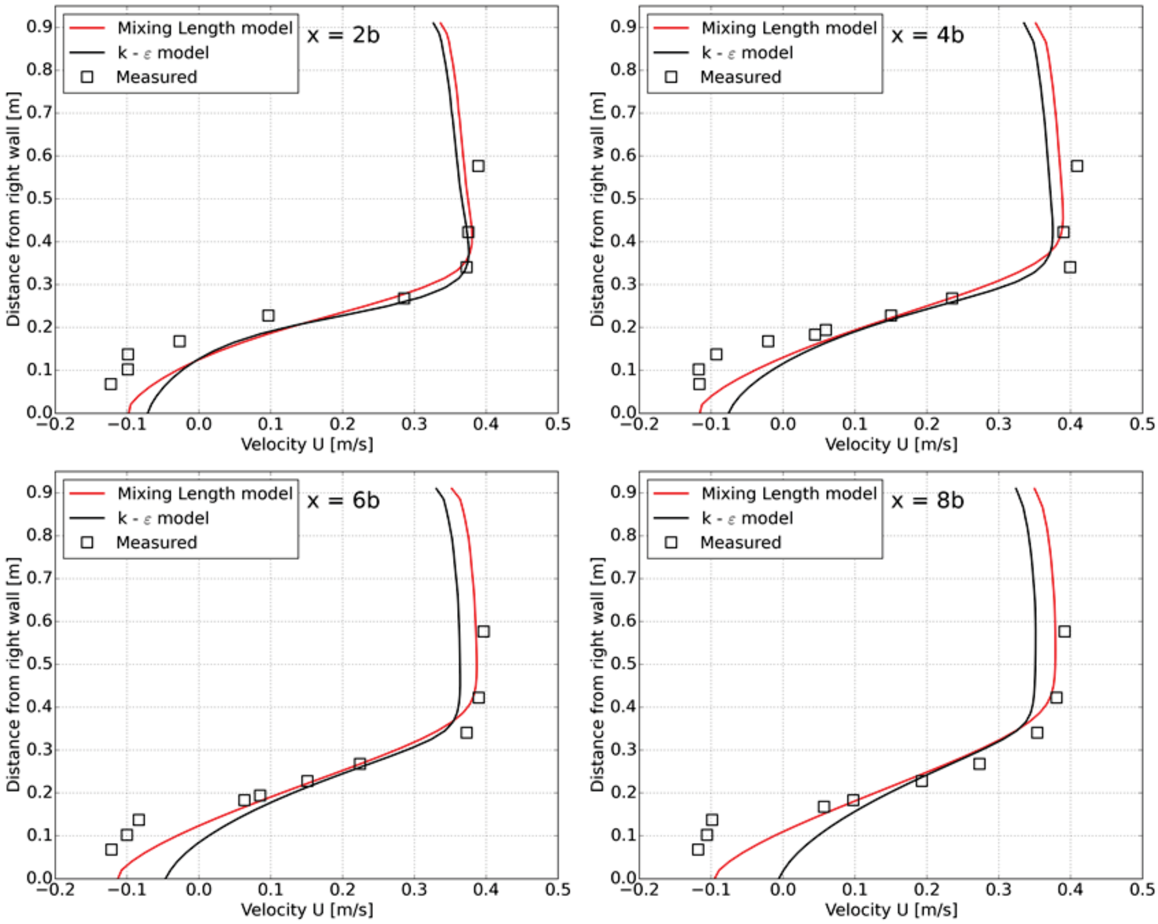

Figure 3. Comparison of the measured and calculated flow velocities in the cross sections x/b = 2, 4, 6 and 8

TABLE I. RMSE VALUES FOR THE k-ε MODEL AND THE MIXING LENGTH MODEL IN ´THE CROSS SECTIONS X/B = 2, 4, 6 AND 8

RMSE [m/s] Velocity U	**x = 2b**	**x = 4b**	**x = 6b**	**x = 8b**
k-ε model	0.0687	0.0644	0.0710	0.0929
Mixing Length model	0.0658	0.0539	0.0470	0.0609

VI. CONCLUSION

This paper describes the depth-averaged Mixing Length turbulence model and its implementation into the open source 2D depth-averaged numerical model Telemac-2D. The zero-equation turbulence model combines the depth-averaged parabolic eddy viscosity model with the Prandtl's mixing length theory for the horizontal in order to account for both the vertical and horizontal turbulence production. The computed eddy viscosity is composed of three components, namely a constant, a bed shear generated term and a transverse shear generated term. Hence the main characteristic is that the turbulence model accounts for the physical influence of the local horizontal velocity gradients on the turbulent eddy viscosity to be computed. The model yields or tends to the parabolic eddy viscosity model if the horizontal depth-averaged velocity gradients vanish or if the turbulence is mainly produced by bed friction, respectively.

The depth-averaged Mixing Length model is verified and validated by means of a laboratory experiment concerning the flow around a spur-dyke and the comparison with the two-equation k-ε turbulence model. The validation reveals the correct implementation of the turbulence model and its applicability for open channel flow computations.

The Mixing Length model can be a viable alternative to the zero-equation turbulence models already available in Telemac-2D especially in cases where the transverse shear might be the dominant turbulence generation mechanism like in flows around structures or flows in reservoirs. The computations using the Mixing Length model are around 20% faster than with the k-ε model. However it should be remembered that the Mixing Length turbulence model, unlike the k-ε model, doesn't account for transport processes of turbulent quantities. In its depth-averaged form the proposed model, like the k-ε model, doesn't account for dispersive transport due to vertical non-uniformities of the mean flow velocities when using the theoretical coefficients emerging from the integration. Therefore depending on the type of flow these coefficients may be seen as tuning coefficients.

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