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PHYSICAL AND MATHEMATICAL MODELLING OF SCOUR[*]

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An overview is presented of physical and mathematical modelling of scour around hydraulic and marine structures. Principal ideas, general features and procedures are given. The paper is organized in two sections: Physical modelling and mathematical modelling. The latter section, the major part of the paper, is further organized in three subsections, mathematical modelling of scour around piers/piles, mathematical modelling below pipelines and mathematical modelling around other structures. Over eighty references were included in the paper.

1 Introduction

When a structure is placed in a hydraulic/marine environment, the presence of the structure will change the flow pattern in its immediate neighbourhood, resulting in one or more of the following processes: the contraction of flow; the formation of a horseshoe vortex in front of the structure; the formation of lee-wake vortices (with or without vortex shedding) behind the structure; the generation of turbulence; the occurrence of reflection and diffraction of waves; the occurrence of wave breaking; and the pressure differentials in the soil that may produce "quick" condition/liquefaction allowing material to be carried off by currents. These changes usually cause an increase in the local sediment transport capacity and thus lead to scour.

Flow and the resulting scour processes may be investigated experimentally in the laboratory (in a small-, medium-, or large-scale facility), the physical modelling; or they may be studied theoretically, using analytical/theoretical or numerical methods, the mathematical modelling.

The purpose of this paper is to give an overview of physical and mathematical modelling of flow and scour processes, describing principal ideas, important features, procedures, shortcomings, etc. The present account is by no means a complete review, considering the vast literature that exists today, particularly on physical modelling of scour. Nevertheless, it will attempt to give a general sense of the subject matter, and also it will give some guidance with regard to the existing work.

2 Physical modelling

The purpose of a physical modelling study is basically to understand scour/flow processes. Tremendous amount of knowledge has accumulated over the years on scour

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around hydraulic and marine structures, and this is mainly due to physical modelling studies. This knowledge is covered extensively by many excellent text books, Breusers and Raudkivi (1991), Hoffmans and Verheij (1997), Melville and Coleman (2000) mainly for scour processes around hydraulic structures, and Herbich (1981), Herbich, Schiller, Watanabe and Dunlap (1984) and Whitehouse (1998), mainly for marine structures. The topic scour around marine structures has also been covered in the recent book by Sumer and Fredsøe (2002). The following paragraphs will give some of the highlights of physical modelling studies of scour.

Laboratory experiments. As mentioned in the preceding paragraphs, the purpose of a physical modelling study is to understand scour/flow processes. To achieve this, we do experiments in the laboratory. These experiments may be actual scour experiments; they may be flow experiments (over a rigid bed or over a sediment bed); or they may be both. As for the **flow experiments**, the idea is to understand flow mechanisms involved in the scour processes, such as the horseshoe vortex process, vortex shedding process, etc. Flow visualization is one of the methods to get an insight into the flow processes. There are several techniques for flow visualization studies such as the dye technique (including the multi-colour dye technique); the milk technique; the technique involved small, light plastic particles, sometimes in combination of laser sheet of light; the hydrogen-bubble technique; and the smoke-wire technique (in the case of wind tunnel). Regarding the flow measurements, these are, among others, (1) velocity measurements, using various techniques such as micro propellers (bi-directional, or otherwise); the hot-wire (or hot-film) technique; Laser Doppler Anemometry (LDA); and Particle Image Velocimetry (PIV), among others, and (2) bed/wall shear stress measurements, using the hot-film technique (Sumer, Arnskov, Christiansen and Jørgensen, 1993) and the shear-plate technique (Kamphuis, 1975, Simons, Grass and Mansour-Tehrani, 1992). Clearly, the space and time resolution in these measurements should be small enough to "resolve" the flow structures involved in the processes. For micro propellers, the measuring-volume size may be of O(0.5 cm). For LDA, the measurement volume size may be O(0.1 mm)×O(0.1 mm)×O(2mm) while, for PIV, it may be as small as O(0.5 mm) (Lara, Cowen and Sou, 2002).

The bed shear stress measurements may prove very useful to study flow/scour processes. The bed shear stress information is also crucial for scour protection. The hot-film technique to measure the bed shear stress involves dimensions like O(0.2 mm)×O(0.8 mm), and the shear plate technique O(25 cm). The hot-film technique cannot be used in the case of rough beds, and therefore the shear plate technique was developed. However, in this latter application, the shear plate has to be large enough to be able to "sense" the tangential force exerting on it. From the sheer size of the shear plate, it is clear that the turbulence resolution in the bed shear stress cannot be achieved in this case.

With regard to the time resolution, this is determined by the time resolution of the instrument itself and also by the sampling frequency of the data acquisition system. Clearly, the time resolution of the measurement should be "fine" enough to be able to resolve the processes in time.

As for the actual **scour experiments**, two main issues are (1) to measure the time evolution of the bed morphology at/near the structures; and (2) if interested only in the equilibrium scour, to measure the equilibrium bed topography only when the scour process attains its equilibrium. Underwater mini (pen size) video cameras prove useful in monitoring the time development of scour processes, both at the structure and in the "near-field" (the latter with the help of thin, measuring pins driven deep into the bed over a mesh area in plan view). Unfortunately, stereo-photography techniques have not been implemented in the laboratory so far, to measure 3-D bed topography (with still pictures, or with sequence of pictures with a reasonable time resolution), largely because of small dimensions involved in the laboratory. (Field applications of such equipment are available, however). In the case of a 2-D scour situation such as 2-D scour below a pipeline, a bed profiler (Christensen, 1981) also can be used to scan the scour profile. In the latter case, or in the case where a plane section will be monitored (e.g., scour in radial planes in pier/pile scour), a laser sheet may be used to monitor the scour, Roulund, Sumer and Fredsøe, 2004). It may also be mentioned that the so-called rotary sonars (rotary fan- and pencil-beam acoustic images are basically acquired in the latter) are applied in the field to obtain bed morphology variations in 3-D (Hay and Speller, 2004), and may be implemented in large-scale laboratory facilities.

Procedure in a physical modelling study. The procedure in a typical physical modelling study (including hydraulic scale model studies) is as follows: (1) Conduct laboratory experiments, directed towards understanding of the hydraulic/hydrodynamic and scour processes; (2) Identify the parameters (normally in non-dimensional forms) responsible for these processes; (3) Once the "governing" parameters are identified, extend your test matrix so as to cover a reasonable coverage of the ranges of the governing parameters; (4) Plot the hydraulic/hydrodynamic quantities from the data as functions of the governing parameters; (5) Give physical explanations for the obtained variations; and plot the data in terms of design/guideline diagrams for the ranges of the governing parameters normally encountered in practice.

Scale effects in physical modelling. An important issue in physical modelling is the so-called scale effects. Scale effects must be considered when the results of small-scale physical-modelling experiments are extrapolated to real-life situations.

There are several scale effects, such as the effect of the Reynolds number; the effect of the roughness (the bed roughness and the roughness on the surface of the structure); the effect of the bed ripples (the structure-size-to-ripple-size ratio is much larger in the field than in the laboratory); the effect of wave/soil interaction; the effect of incoming-flow turbulence; the effect of structural aspects (for example, in the case of scour below pipelines, the pipe stiffness is an important factor for 3-D scour).

The influence of scale effects (in general) when investigating processes in the laboratory have been discussed by, amongst others, Hughes, (1993), Oumeraci (1994); Whitehouse (1998), Sutherland and Whitehouse (1998) and Sumer, Whitehouse and Tørum (2001). Clearly no one experimental set up can meet the scaling requirements for all situations and hence each scenario will have its own scaling solution.

The following discussion addressing the situation for scour experiments draws from Sumer et al. (2001). Whilst the scaling of waves and currents in the laboratory is well understood in practice (Froude and Reynolds scaling), the many requirements of an experiment preclude an exact scaling exercise. Obviously scale effects can be reduced to a minimum by running tests in suitably large facilities. In order to design the experiment at an appropriate scale the most important factor is to have an understanding of the important processes acting in the prototype situation. This means that the model scaling can be optimized to address the influence of these processes. Once the relevant processes are determined, the next step will be to determine appropriate and meaningful non-dimensional quantities to represent these processes. Obviously, these nondimensional quantities in the laboratory experiments need to be maintained in the same range as that experienced in the field. If necessary, the influence of scale effects can be examined by running tests at a number of scales to understand better the prototype situation (a sensitivity study).

The great advantage of physical modelling is that the key factors causing scour development can be investigated in a controlled fashion. The results can be represented by non-dimensional semi-empirical parameters that can be used in design work. However, the potential influence of scale effects or model effects must always be considered in the interpretation of results.

3 Mathematical modelling

3.1. *Mathematical modelling of scour around piers/piles*

Mathematical models for flow/scour around a pier/pile may be divided in two categories: Simple models and advanced models.

As for the **simple models**, there are excellent models, which could achieve a fairly substantial amount of information, useful for engineering calculation/assessments with regard to various aspects of the scour process. Of these, the following can be mentioned. Dey and Bose (1994) developed a model to calculate the bed shear stress in an equilibrium scour hole around the base of a circular pier. Dey, Bose and Sastry (1995), in another study, developed a 3-D kinematic model for vortex flow around a circular pier in an equilibrium scour hole under a clear water scour regime. Dey (1996), in a follow-up study, presented a model to estimate the temporal variation of sediment pick-up for the same scour regime. Tsujimoto (1986) gave a model for scour around a pier, using sediment pick-up concept. More recently, Dey (1999) presented a model to estimate the time variation of scour depth in clear-water and live-bed scour regimes. Miller and Sheppard (2002) presented a model for scour around a circular pier, with the aim of developing a prediction tool to determine the rate at which scour occurs. The model was developed for clear-water conditions, but it can be extended to the live-bed regime.

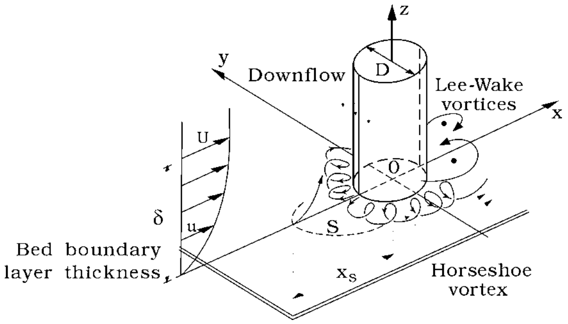

Figure 1. Description sketch. Flow around a pier/pile. S: Separation line.

The horseshoe vortex is an essential part of the 3-D flow around a slender pile/vertical cylinder (Fig. 1). In recent years, the numerical treatment of the horseshoe vortex has been made by several researchers. A comprehensive review of the work until early nineties is given by Deng and Piquet (1992). 3-D numerical calculations of the flow around a vertical, wall-mounted cylinder have also been carried out with the purpose of studying the scour, the **advanced models**. The following paragraphs will give a brief, partial account of advanced models.

In the case of **steady currents**, Olsen et al. (1993, 1998) obtained steady solutions. They resolved the horseshoe vortex, and in their morphology calculations, they simulated clear-water scour. Richardson and Panchang (1998) also obtained steady solutions. They resolved the horseshoe vortex. However, no scour simulation was undertaken. Instead, three frozen bed topographies, each representing snap shot of the scoured bed during the scour process, were represented. Tseng et al. (2000), on the other hand, obtained transient solutions, simulated a rigid plane-bed flow, and they resolved the horseshoe vortex and vortex shedding (Fig. 1). No scour simulation was undertaken. They inferred, however, some conclusions for scour from the rigid bed simulation. Nurtjahyo, Chen, Briau, Li and Wang (2002) also obtained steady solutions (for a rectangular pier). They resolved the horseshoe vortex. No scour simulation was undertaken. Chen (2002) achieved flow calculations and scour simulations for an array of circular and rectangular bridge piers. Horseshoe vortex in front of the piers was resolved. Scour calculations were made for cohesive soil, a relatively easy case compared with non-cohesive soil such as sand and silt. Nevertheless, the dramatic effect of the flow has been illustrated beautifully in Chen's (2002) three-dimensional computer imagery. Roulund et al. (2002 and 2004) obtained steady solutions for a circular pier. They resolved the horseshoe vortex. They also achieved transient solutions in some cases with the vortex shedding resolved). However, they achieved scour calculations (non-cohesive soil, sand) with the steady solution to avoid prohibitively large computational times. We shall return to Roulund et al.'s study later in the section.

In the case of **waves**, only two studies are available. Kobayashi (1992) made an attempt to simulate the 3-D flow around a circular pile in the case of an oscillatory flow, using the so-called vortex-segment model. This model is basically an extension of the familiar 2-D discrete-vortex model (see Sumer and Fredsøe, 1997, for a review of discrete vortex models) to the 3-D case. No horseshoe vortex and no vortex shedding were obtained, since the KC number was below the critical values for these flow features to exist. However, the model did produce the unsteady behaviour of the lee-wake vortices. Yuhi, Ishida and Umeda (2000) have achieved a 3-D numerical solution of the N.-S. equations for both steady currents and oscillatory flows. These numerical calculations were undertaken for two kinds of beds: for a rigid, plane bed and for a rigid, truncated-cone-shaped bed (simulating a scoured bed) in the case of the steady current; and for a rigid, plane bed in the case of the oscillatory flow. The calculations are for laminar-regime flows.

The **general approach on numerical simulation of scour** around a pier/pile adopted in the previously mentioned work could possibly be best described by reference to the work of Roulund et al. (2002 and 2004).

Roulund et al. (2002, 2004) have simulated the flow around a pile, using a 3-D code, EllipSys3D, a three-dimensional general-purpose flow solver developed at Risø National Laboratory, Denmark and Department of Energy Engineering at the Technical University of Denmark (Sørensen, 1995, and Michelsen, 1992). The code solves the 3-D N.-S. equations, the Reynolds-averaged Navier-Stokes equations

$$\frac{\partial \rho U_i}{\partial t} + \frac{\partial \rho U_i U_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[(\mu + \mu_T)\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right] \qquad (1)$$

in which U_i is the i-th component of velocity; t is the time; x_i is the Cartesian coordinates; ρ is the fluid density; p is the dynamic pressure; μ is the viscosity; and μ_T is the eddy viscosity, calculated by a two-equation eddy-viscosity type turbulence model. It is basically a multiblock finite volume discretisation of the Reynolds Averaged Navier-Stokes equations. A variety of turbulence models are available. The model is under constant development. It has been implemented successfully in various engineering problems such as those in wind engineering and aeronautical engineering. The basic principles of the model have been described in Sørensen, 1995 and Michelsen, 1992. The k-ω turbulence model (Wilcox, 1993, and Menter, 1992) was used for closure. The latter closure model was adopted because of its better performance for flows with strong pressure gradients. The k-ω model is essentially based on two equations, one equation for k and the other for ω in which k is the turbulent kinetic energy and ω is the specific dissipation of turbulent kinetic energy:

$$k = \frac{1}{2}\overline{u_i' u_i'} \quad and \quad \omega = \frac{\varepsilon}{k\beta^*} \qquad (2)$$

in which u_i' is the fluctuating component of the velocity and ε the dissipation of turbulent kinetic energy and β^* is one of the model closure constants. The equation for k is

$$\frac{\partial \rho k}{\partial t}+\frac{\partial \rho U_j k}{\partial x_j}-\frac{\partial}{\partial x_j}\left[(\mu+\sigma_k \mu_T)\frac{\partial k}{\partial x_j}\right]=\tau_{ij}\frac{\partial U_i}{\partial x_j}-\beta^* \rho k\omega \qquad (3)$$

and that for ω

$$\frac{\partial \rho \omega}{\partial t}+\frac{\partial \rho U_j \omega}{\partial x_j}-\frac{\partial}{\partial x_j}\left[(\mu+\sigma_\omega \mu_T)\frac{\partial \omega}{\partial x_j}\right]=\frac{\gamma}{\nu_T}\tau_{ij}\frac{\partial U_i}{\partial x_j}-\beta\rho\omega^2+2\rho(1-F_1)\frac{\sigma_{\omega 2}}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}$$
(4)

In the preceding equations, τ_{ij} is the Reynolds stress and σ_k, σ_ω, γ and β are expressions given in terms of model constants and the so-called blending function. The explicit expressions for the latter will not be given here but can be found in Roulund et al. (2004).

There are three versions of *k*-ω model in the literature: (1) the original *k*-ω model (Wilcox, 1993); (2) the *k*-ω, BSL model; and (3) the *k*-ω, SST model. The latter two versions were developed by Menter (1993) to improve Wilcox's original model so that an even higher sensitivity can be obtained for adverse-pressure-gradient flows. Menter (1993) made an extensive comparison between (1) the classic *k*- ε model; (2) the original *k*-ω model; (3) the *k*-ω, BSL model; and (4) the *k*-ω, SST model for various well documented flows. The tested flows were, among others, two kinds of adverse-pressure flow (one having a very strong adverse pressure gradient, so strong that separation occurs); the backward-facing-step flow; and the flow past a NACA 4412 airfoil at an angle of attack near maximum lift condition. This inter-comparison exercise revealed that the *k*-ω, SST model gave the most accurate results while the *k*- ε model did not yield as accurate results as the other three for the tested adverse-pressure-gradient flows. The *k*-ω, SST model has been adopted in the implementation of the model in Roulund et al. (2002, 2004).

For steady-state flow calculations, the SIMPLE algorithm (Patankar, 1980) is used. In this algorithm, the pressure field is calculated and the velocity field is corrected so that the continuity equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho U_j}{\partial x_j}=0 \qquad (5)$$

is satisfied in an iterative manner. By under-relaxation of the correction to the velocity field, the transient components of the flow are suppressed. Although the majority of the present study involved steady-state flow calculations, some transient flow calculations have also been performed, as mentioned previously

Roulund et al.'s scour calculations involved a morphologic model that couples the flow solution with a sediment transport description, and routines for updating the computational mesh based on the mass balance of sediment. There are three elements in the morphologic model: (1) the bed load; (2) the sand slide; and (3) the equation of continuity for sediment. A two-dimensional bed load description has been developed. This description is actually an extension of the bed-load equation of Engelund and Fredsøe (1976) to a 2-D vectorial representation. With regard to the sand slide, observations show that, during the development of the scour process, there are areas at the upstream side of the scour hole where the local bed slope exceeds the angle of repose, and, as a result, shear failures occur at these locations. Two "ingredients" of this latter process are that (1) the backward flow at the base of the pile erodes the foot of the upstream slope of the scour hole, and (2) there is a continuous sediment supply into the scour hole from upstream. As a result, the bed avalanches when the slope exceeds the angle of repose. Roulund et al. have incorporated this aspect in the code, in the form of sediment transport. Finally the equation of continuity for sediment involves the mass balance for sediment at each grid point on the bed

$$\frac{\partial h}{\partial t}=\frac{-1}{1-n}\frac{1}{A}\sum_{i=1}^{4}(\overrightarrow{q_{b,i}}\bullet\overrightarrow{n_i})|l_i| \tag{3}$$

in which h is the bed elevation, n is the porosity (taken as n = 0.4 in the present calculations); A is the projected area (on the x,y plane, the plan view) of a small bed element; i indicates the number assigned to each side of the projected area (i=1,..4); $\overrightarrow{n_i}$ are the normal vector at the i–th side of the projected area, $\overrightarrow{q_{b,i}}$ is the sediment transport vector at the i–th side of the projected area; and $|l_i|$ is the length of the i–th side of the bed element.

The procedure in the computations was as follows: (1) Generate the mesh; (2) Calculate the flow; (3) Calculate the sediment transport due to bed load; (4) Update the bed; (5) Check the sand slide; and (6) Return Step 1. The morphological time step in the calculations was 0.02 s initially, and gradually raised to 0.1 s. The computational time was 2.5 months on an Alpha 21264 workstation, equivalent to a 1500 MHz Pentium IV.

Finally, Roulund et al. (2002, 2004) note that sediment transport is taken as the bed load alone (no suspended-load sediment transport is considered). They indicate that data reported in Baker (1986) (where the velocity is increased gradually so as to cover the flow regimes from the clear-water scour to the live-bed scour with the bed-load, the bed-load and suspended load and finally the suspended-load mode sheet-flow regimes; see Melville and Sutherland, 2000, p. 493, or Sumer and Fredsøe, 2002, p. 179) suggests that this effect is not radically significant.

A detailed validation of the model against experiments has been undertaken for the flow model for the case where the bed was plane and rigid, Roulund et al. (2002, 2004), and it has been concluded that the model captures fairly well the mean features of the 3-D flow, including the horseshoe vortex (one of the key elements in the scour process), meaning that it can be used for the calculation of the scour process around a pile placed in a sediment bed.

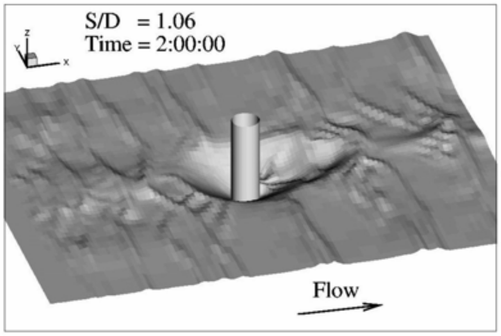

Figure 2. Scour hole in the equilibrium stage obtained in Roulund et al.'s (2002, 2004) numerical simulation

In the numerical simulation of scour, vortex shedding was not resolved (the steady-state flow simulation was adopted), as already pointed out to avoid the large computational times required for the transient flow solution. The water depth in the simulation was δ =20 cm, corresponding to the boundary layer thickness. The pile diameter was D = 10 cm. The mean flow velocity was V = 46 cm/s. The Shields parameter was θ = 0.11, meaning that the scour was in the live bed regime. In the simulation, the bed ripples were resolved. Fig. 2 shows the scour picture at the equilibrium stage.

It may be noted that the previously mentioned model has also been implemented to simulate the flow around and forces on a sphere placed near a bed.

Finally, scour around a large pile in waves (scour caused by the steady-streaming effect) has been treated theoretically by several investigators. Saito, Sato and Shibayama (1990), Saito and Shibayama (1992) along with Katsui (1992) and Katsui and Toue (1993) made attempts to simulate the scour process numerically for the case of a circular pile, and Kim, Iwata, Miyaike and Yu (1994) for the case of two circular piles.

3.2. *Mathematical modelling of scour below pipelines*

The mathematical models on scour below pipelines may be divided into three categories: (1) the potential-flow models, (2) the advanced models, and (3) the integrated models. The first two models concern the flow and the resulting scour for a fixed pipeline, while the third one aims at predicting the time evolution of the interaction between a pipeline and the bed under a given time series of wave and current conditions.

Several authors have developed **potential-flow models** for flow and scour around pipelines, Chao and Hennessy (1972), Chiew (1991), Hansen, Fredsøe and Mao (1986), Hansen (1992), Bernetti, Bruschi, Valentini and Venturi (1990), Li and Cheng (1999 a). An important "constraint" with the potential-flow models is that they cannot handle the lee-wake flow, and therefore they cannot handle the lee-wake erosion, an important stage in the scour process. Another constraint is that some of the models assume that the bed shear stress eventually attains the critical value of the bed shear stress. This assumption may be valid only for the clear-water scour case. For the live-bed case, the bed shear stress eventually attains the level of the bed shear stress in the approach flow, which is necessarily larger than the critical bed shear stress. A detailed account of these models can be found in Sumer and Fredsøe (2002, Chapter 2).

Regarding the **advanced models**, these models calculate the flow around the pipe, solving basically the Navier-Stokes (N.-S.) equations numerically, similar to that described for pier/pile scour in the previous section. Numerical treatment of flow around *free cylinders* has improved significantly with the increasing capacity of computers. A detailed review of the subject (in the case of a free cylinder) can be found in Sumer and Fredsøe (1997, Chapter 5). While there have been numerous numerical investigations of flow around free cylinders, there have been comparatively few numerical investigations of flow around pipelines (where the pipe is confined with a rigid or an erodible bed below), Leeuwenstein and Wind (1984); van Beek and Wind (1990), Brørs (1999) (Fig. 3) and Smith and Foster (2004), involving k-ε simulations; Li and Cheng (1999 b, 2001 and 2002), and Heather and Foster (2004) involving Large Eddy Simulations (LES); and Sumer et al. (1988), Jensen, Jensen, Sumer and Fredsøe (1989); and Jensen et al. (1990), involving discrete-vortex models. These studies, in the case of the erodible bed, also involve the numerical calculation of the bed morphology. Recently Liang, Cheng and Li (2004) have made an intercomparison study of the performance of various turbulence models (k-ε, k-ω and Smagorinski's subgrid scale (SGS) models) in conjunction with scour around a pipeline in steady current, a benchmark case with its 2-D simple geometry and a case where there exists a fair amount of experimental data for flow and scour. In this two-part paper, Part 1 summarizes the results of the flow calculations while Part 2 summarizes the results of the scour calculations. Finally, Liang and Cheng (2004 a) have developed a numerical study of scour below a pipeline exposed to waves (Fig. 4). The k-ω closure has been implemented. Both suspended and bed loads of sediment transport have been taken into account in the model. Again, a detailed account of the advanced models can be found in Sumer and Fredsøe (2002, Chapter 2).

As for the **integrated models**, the principal idea here is to develop a computer model which would enable the engineer to predict the emergence and disappearance of scour along a pipeline, the pipeline self-burial, the trench backfilling, migrating sand wave exposure, and undermining of pipelines. Such a model needs to accommodate all the possible processes such as the onset of scour, the tunnel erosion, the wake erosion, the 2-D scour, the 3-D scour, the sagging of the pipeline, the self-burial (in free span areas and at span shoulders), and other sediment transport processes. The input parameters will be the time series of the wave and current climate, the geometrical and structural properties of the pipeline and the soil properties. The output will be the time series of the position of the pipe (with respect to the bed) at every point along the length of the pipeline. Papers by Staub and Bijker (1990), and Hansen, Klomp, Smed, Chen, Bijker and Bryndum (1995) present the results of a work where such an approach has been adopted. It may also be noted that the work by Bernetti et al. (1990) may be considered as one of the first examples of these types of models although it does not include the processes such as the

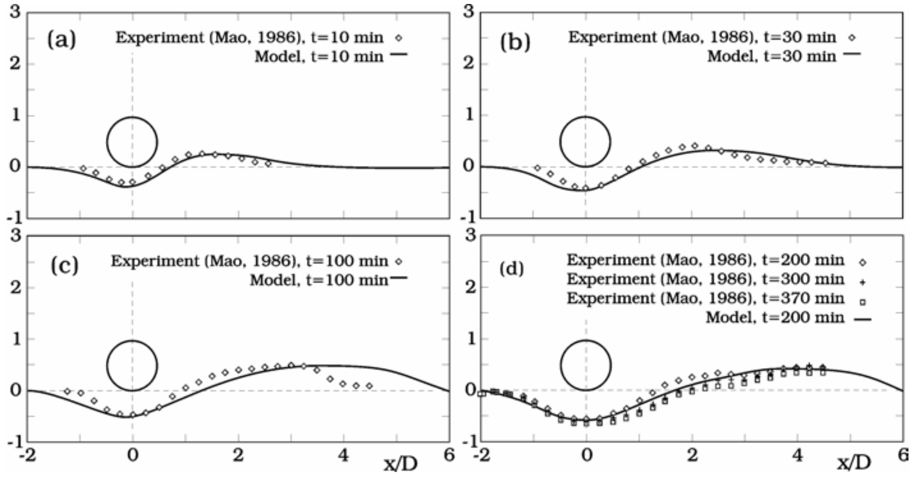

Figure 3. Time development of scour below a pipeline. Numerical model: Brør's (1999) k-ε model. Steady current. Taken from Sumer and Fredsøe (2002).

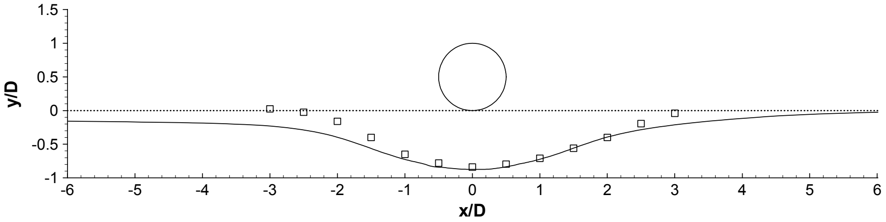

Figure 4. Scour below a pipeline. Equilibrium. Numerical model: Liang and Cheng's (2004 a) k-ω model. Experiment: Sumer and Fredsøe (1990). Waves. Keulegan-Carpenter number, *KC* = 36. By courtesy of Liang Cheng, University of Western Australia.

onset of scour, the wake erosion, and the self-burial (in free span areas and at span shoulders). Finally, the paper by Bruschi, Drago, Venturi, Jiao and Sotberg (1998) gives a detailed account of the existing models to help estimate integrity of free spans.

3.3. *Mathematical modelling of scour around other structures*

This subsection will give a partial account of advanced mathematical modelling studies of flow/scour around hydraulic and marine structures other then the previously mentioned “bench mark” structures.

Groins in streams. When a groin is placed in a stream, the flow will be subject to substantial changes: (1) A “half” horseshoe vortex will be formed in front of the groin; (2) A vortex flow (consisting of (a) vortices shed from the groin, and (b) a re-circulating cell) will be formed at the lee-side of the groin, and (3) the streamlines will contract at the flow side of the groin. The overall effect of these changes is to increase the local sediment transport in the case of an erodible bed, resulting in a local scour/deposition around the groin.

Although there is a substantial amount of work on flow around groins in streams, there are only a few studies investigating the 3-D flow, using advanced flow models. Tingsanchali and Maheswaran (1990) made a 2-D, depth-averaged, k-ε flow computation for a vertical-wall groin. Although it was an important step forward, the depth-averaged flow code could not resolve the horseshoe vortex and other 3-D features of the flow. Mayerle et al. (1995) adopted a 3-D flow code incorporated with several eddy-viscosity models. However, attention was not concentrated on the near-field flow, and therefore it is not known if the model was able to resolve the horseshoe vortex and other 3-D features. Ouillon and Dartus (1997) made a 3-D, k-ε computation of flow around a vertical-wall groin using a 3-D Reynolds solver, which apparently resolved the horseshoe vortex (Ouillon and Dartus, 1997, Fig. 7). The previous studies considered the case of the vertical-wall groin. Miller, Roulund, Sumer, Fredsøe, Truelsen and Michelsen (2003) attempted to address the issue of the effect of the sloping side walls, with the aid of a 3-D flow code. Some early results are presented. The latter authors used the k-ω turbulence model. They considered two kinds of groins: (1) A vertical-wall groin, and (2) A groin with a side slope. Steady-flow simulations were conducted. The horseshoe vortex in front of the vertical-wall groin was resolved. The vortex shedding at the head is not resolved because no transient flow simulations were conducted. Comparison between the vertical-wall groin case and the case of a groin with a side slope indicated that the flow around the groin undergoes notable changes.

Breakwaters. A standing wave that forms in front of a breakwater generates a field of steady streaming, a system of recirculating cells (consisting of bottom and top cells). The formation of the bottom cells is related to the boundary layer over the bed. The sediment on the bed essentially responds to these recirculating cells. The end result is that a scour/deposition pattern in front of the breakwater emerges in the form of alternating scour and deposition areas lying parallel to the breakwater.

Arneborg, Hansen and Juhl (1995 a) developed a numerical model to simulate this 2-D scour process. The model has two modules: the flow module, and the sediment module. The flow module considers the flow in two regions: the boundary layer over the bed; and the flow outside the boundary layer. For the flow outside the boundary layer, the model uses the linear wave theory. The effect of a varying bed geometry is taken into account by making a conformal mapping of the linear solution for a plane bed into the domain with this varying bed geometry. For the boundary layer over the bed, the integrated momentum equation is used, similar to Fredsøe (1984). The model has been developed for a rough bed. In a follow-up study, the same authors studied scour at the trunk section of a rubble-mound breakwater. They viewed that the difference between the vertical-wall breakwater case and the rubble-mound breakwater case lies in the reflection coefficient associated with the rubble-mound structure. The scour depth decreases with decreasing values of the reflection coefficient, as expected. A detailed review of these two models can be found in Sumer and Fredsøe (2002).

In a recent study (Gislason, Fredsøe and Sumer, 2003). the time-dependent 3-D Navier-Stokes (N.-S.) equations with fully non-linear free-surface boundary conditions are solved for the simulation of two-dimensional periodic standing waves in front of a vertical-wall breakwater, and partially standing waves in front of a breakwater with a sloping wall. The model coupled with a sediment transport module has allowed for the simulation of the near-field morphodynamics (scour/deposition). The steady-streaming velocity within the laminar/turbulent wave boundary layer was calculated. The time development of the bed morphology, initially being flat, was simulated until the equilibrium profile is reached. The results in the case of a vertical-wall breakwater reveal the well-known alternating scour/deposition pattern in front of the breakwater.

Sea walls. Seawalls are structures placed parallel to the shoreline, to separate a land area from a water area. In the case of non-breaking waves, the situation will be rather similar to that described for vertical-wall breakwaters (see the previous subsection). In the case of breaking waves, the breaking process creates strong downward directed flows (in the form of a jet) that erode the bed, and mobilizes the sediment at the toe of the structure. These processes will presumably lead to scour at the seawall.

McDougal, Kraus and Ajiwibowo's (1996) carried out a numerical study of scour in front of a vertical seawall. Their numerical model basically comprises the following two elements: (1) a wave-transformation model, to predict the wave transformation (refraction, shoaling, breaking, and reflection from the seawall), and (2) a cross-shore sediment transport algorithm, to predict the beach profile. The model, widely known as SBEACH (Larson and Kraus, 1989), has been extended to include the effect of reflection at the seawall as well. An empirical expression has been determined from the obtained numerical scour-depth data.

A model similar to that of McDougal et al. (1996) has been developed by Rakha and Kamphuis (1997 b) to predict the scour in the vicinity of a seawall. The model essentially comprises four modules, namely the wave-transformation module (Rakha and Kamphuis, 1994), the wave-induced-current module (Rakha and Kamphuis, 1997 a), the sediment-transport module (Rakha and Kamphuis, 1997 b), and the morphology module (Rakha and Kamphuis, 1997 b).

Sumer and Fredsøe (2002) can be consulted for a detailed account of the subject.

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